

CONDITIONAL MEASURES OF DETERMINANTAL POINT PROCESSES

ALEXANDER I. BUFETOV

ABSTRACT. For a class of one-dimensional determinantal point processes including those induced by orthogonal projections with integrable kernels satisfying a growth condition, it is proved that their conditional measures, with respect to the configuration in the complement of a compact interval, are orthogonal polynomial ensembles with explicitly found weights. Examples include the sine-process and the process with the Bessel kernel. The argument uses the quasi-invariance, established in [1], of our point processes under the group of piecewise isometries of \mathbb{R} .

1. FORMULATION OF THE MAIN RESULT.

1.1. Conditional measures. Let E be a locally compact complete metric space, let $\text{Conf}(E)$ be the space of configurations on E . Given a configuration $X \in \text{Conf}(E)$ and a subset $C \subset E$, we let $X|_C$ stand for the restriction of X onto the subset C .

A point process on E is a Borel probability measure on $\text{Conf}(E)$. For such a measure \mathbb{P} , the measure $\mathbb{P}(\cdot|X; C)$ on $\text{Conf}(E \setminus C)$ is defined as the conditional measure of \mathbb{P} with respect to the condition that the restriction of our random configuration onto C coincides with $X|_C$. More formally, consider the surjective restriction mapping $X \rightarrow X|_C$ from $\text{Conf}(E)$ to $\text{Conf}(C)$. Fibres of this mapping can be identified with $\text{Conf}(E \setminus C)$, and conditional measures, in the sense of Rohlin [6], are precisely the measures $\mathbb{P}(\cdot|X; C)$. If the point process \mathbb{P} admits correlation measures of order up to l , then, given distinct points $q_1, \dots, q_l \in E$, we let $\mathbb{P}^{q_1, \dots, q_l}$ stand for the l -th reduced Palm measure of \mathbb{P} conditioned at points q_1, \dots, q_l (here and below we follow the conventions of [1] in working with Palm measures).

The main results of this note can informally be summarized as follows. If the measure $\mathbb{P}(\cdot|X; C)$ is supported on the subspace of configurations with precisely l particles and the reduced Palm measures, conditioned at different l -tuples of points, are equivalent, then, under certain additional assumptions (see Proposition 3.1 below), the conditional measure $\mathbb{P}(\cdot|X; C)$ has the form

$$Z^{-1}(q_1, \dots, q_l) \frac{d\mathbb{P}^{p_1, \dots, p_l}}{d\mathbb{P}^{q_1, \dots, q_l}}(X|_C) d\rho_l(p_1, \dots, p_l),$$

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where q_1, \dots, q_l is almost any fixed l -tuple, ρ_l is the l -th correlation measure of \mathbb{P} and $Z(q_1, \dots, q_l)$ is the normalization constant. In particular, for one-dimensional determinantal processes induced by projections with integrable kernels satisfying a growth condition and C the complement of a compact interval, it is proved that $\mathbb{P}(\cdot|X; C)$ is an orthogonal polynomial ensemble with the weight found explicitly. We proceed to precise formulations.

Given a compact subset $B \subset E$ and a configuration $X \in \text{Conf}(E)$, let $\#_B(X)$ stand for the number of particles of X lying in B . Given a Borel subset $C \subset E$, we let \mathcal{F}_C be the σ -algebra generated by all random variables of the form $\#_B, B \subset C$. Write $\mathcal{F}_C^{\mathbb{P}}$ for the \mathbb{P} -completion of \mathcal{F}_C .

Definition (Ghosh and Peres [3], [4]). A point process \mathbb{P} on E is called *rigid* if for any compact subset $B \subset E$ the function $\#_B$ is $\mathcal{F}_{E \setminus B}^{\mathbb{P}}$ -measurable.

For a subset $C \subset E$ and a natural number l , we write $\text{Conf}_l(C)$ for the space of l -particle configurations on C ; in other words, the space of all subsets of C of cardinality l . Rigidity implies that for any precompact set $B \subset E$ and \mathbb{P} -almost any X the conditional measure $\mathbb{P}(\cdot|X; E \setminus B)$ is supported on the subset $\text{Conf}_l(B)$, where $l = \#_B(X)$.

Let $U \subset \mathbb{R}$ be an open set endowed with the Lebesgue measure Leb . Let $\Pi(x, y), x, y \in U$, be a kernel smooth in the totality of variables. Assume that the kernel Π induces an operator of orthogonal projection acting in $L_2(U, \text{Leb})$; slightly abusing notation, we keep the same symbol Π for this operator. Let L be the range of Π . By the Macchi-Soshnikov Theorem, the operator Π induces a determinantal measure \mathbb{P}_{Π} on $\text{Conf}(U)$.

For the sine-process, rigidity is due to Ghosh [3]; for the determinantal point processes with the Airy and the Bessel kernel, rigidity has been established in [2].

For $p \in U$, set $L(p) = \{\varphi \in L : \varphi(p) = 0\}$ and let Π^p be the operator of orthogonal projection onto $L(p)$. By the Shirai-Takahashi Theorem [7], the determinantal measure \mathbb{P}_{Π^p} induced by the operator Π^p is the reduced Palm measure of \mathbb{P}_{Π} at the point p : $\mathbb{P}_{\Pi^p} = \mathbb{P}_{\Pi}^p$.

Assumption 1. Let $p \in U$. If $\varphi \in L$ is such that $\varphi(p) = 0$, then $\frac{\varphi(x)}{x - p} \in L$.

Proposition 3.3 in [1] shows that Assumption 1 holds, in particular, for kernels Π having integrable form $\Pi(x, y) = \frac{A(x)B(y) - B(x)A(y)}{x - y}$.

1.2. The trace-class case. In the first theorem, we will make a restrictive

Assumption 2. We have $\int_U \frac{\Pi(x, x)dx}{1 + |x|} < +\infty$.

The Bessel kernel satisfies Assumption 2. Under Assumption 2, the operators $(|x|+1)^{-1}\Pi$ and $(x+i)^{-1}\Pi$ belong to the trace class, and for $p, q \in U$, the multiplicative functional

$$(1) \quad \Psi_{p,q}^\Pi(X) = \prod_{x \in X} \left(\frac{x-p}{x-q} \right)^2$$

exists and belongs to $L_1(\text{Conf}(U), \mathbb{P}_{\Pi^q})$. By Corollary 4.12 in [1], we have

the \mathbb{P}_{Π^q} -almost sure equality $\frac{d\mathbb{P}_{\Pi^p}}{d\mathbb{P}_{\Pi^q}} = Z_{p,q}^{-1} \Psi_{p,q}^\Pi$, where $Z_{p,q}$ is the normaliza-

tion constant. Since, for $p, q, r \in U$, we have $\frac{d\mathbb{P}_{\Pi^p}}{d\mathbb{P}_{\Pi^q}} = \frac{d\mathbb{P}_{\Pi^p}}{d\mathbb{P}_{\Pi^r}} \frac{d\mathbb{P}_{\Pi^r}}{d\mathbb{P}_{\Pi^q}}$, there exists

a positive function $\rho^\Pi : U \rightarrow \mathbb{R}$ such that $\frac{\Pi(p,p)}{\Pi(q,q)} \frac{d\mathbb{P}_{\Pi^p}}{d\mathbb{P}_{\Pi^q}} = \frac{\rho^\Pi(p)}{\rho^\Pi(q)} \Psi_{p,q}^\Pi$, or, equivalently, that

$$(2) \quad \int_{\text{Conf}(U)} \Psi_{p,q}^\Pi(X) d\mathbb{P}_{\Pi^q}(X) = \frac{\rho^\Pi(q)}{\rho^\Pi(p)} \frac{\Pi(p,p)}{\Pi(q,q)}.$$

If Π is the Christoffel-Darboux kernel of a family of orthogonal polynomials and \mathbb{P}_{Π} the corresponding orthogonal polynomial ensemble, then ρ^Π is the weight. The function ρ^Π is defined up to a multiplicative constant.

Theorem 1.1. *Let $U \subset \mathbb{R}$ be an open set. Let $\Pi(x, y)$, $x, y \in U$, be a smooth kernel that induces an operator of orthogonal projection acting in $L_2(U, \text{Leb})$, satisfying Assumptions 1, 2 and such that the determinantal point process \mathbb{P}_{Π} is rigid. Let $I \subset U$ be a compact interval. Then*

1. *For almost any $2l$ distinct points $p_1, \dots, p_l, q_1, \dots, q_l \in U$, we have the $\mathbb{P}^{q_1, \dots, q_l}$ -almost sure equality*

$$\frac{d\mathbb{P}^{p_1, \dots, p_l}}{d\mathbb{P}^{q_1, \dots, q_l}}(X) = \frac{\det \Pi(q_i, q_j)|_{i,j=1, \dots, l}}{\det \Pi(p_i, p_j)|_{i,j=1, \dots, l}} \prod_{1 \leq i < j \leq l} \left(\frac{p_i - p_j}{q_i - q_j} \right)^2 \prod_{i=1}^l \frac{\rho^\Pi(p_i)}{\rho^\Pi(q_i)} \Psi_{p_i, q_i}^\Pi(X);$$

2. *For \mathbb{P}_{Π} -almost any $X \in \text{Conf}(U)$, the measure $\mathbb{P}(\cdot | X; U \setminus I)$ has the form*

$$(3) \quad Z(I, X)^{-1} \prod_{1 \leq i < j \leq \#_I(X)} (t_i - t_j)^2 \prod_{i=1}^{\#_I(X)} \rho_{I,X}^\Pi(t_i),$$

where $Z(I, X)$ is the normalization constant and the function $\rho_{I,X}^\Pi$ satisfies, for any $p, q \in I$, the relation

$$(4) \quad \frac{\rho_{I,X}^\Pi(p)}{\rho_{I,X}^\Pi(q)} = \frac{\rho^\Pi(p)}{\rho^\Pi(q)} \prod_{x \in X \setminus I} \left(\frac{x-p}{x-q} \right)^2.$$

Remark. The order of the points in Claim 1 is immaterial: for any permutation π on l symbols, by definition, we have $\prod_{i=1}^l \Psi_{p_i, q_i}^\Pi = \prod_{i=1}^l \Psi_{p_i, q_{\pi(i)}}^\Pi$.

Let $U = (0, +\infty)$, take $s > -1$ and consider the Bessel kernel

$$(5) \quad J_s(x, y) = \frac{\sqrt{x} J_{s+1}(\sqrt{x}) J_s(\sqrt{y}) - \sqrt{y} J_{s+1}(\sqrt{y}) J_s(\sqrt{x})}{2(x - y)}$$

(see, e.g., page 295 in Tracy and Widom [14]). The kernel J_s induces on $L_2((0, +\infty), \text{Leb})$ the operator of orthogonal projection onto the subspace of functions whose Hankel transform is supported in $[0, 1]$ (see [14]).

Proposition 1.2. *For any $s > -1$, we have $\rho^{J_s}(t) = t^s$.*

1.3. The Hilbert-Schmidt Case. We now impose a weaker

Assumption 3. *We have $\int_U \frac{\Pi(x, x) dx}{1 + x^2} < +\infty$.*

It follows that the operator $(x + i)^{-1} \Pi$ is Hilbert-Schmidt. The sine-kernel, for example, satisfies Assumption 3 but not Assumption 2.

Let $\lambda(x)$ be a continuous function on \mathbb{R} satisfying

$$(6) \quad \sup_{x \in \mathbb{R}} |x^2 \lambda(x) - x| < +\infty.$$

For example, one can take $\lambda(x) = (x + i)^{-1}$ or $\lambda(x) = \frac{x}{x^2 + 1}$.

We start by formulating an auxiliary

Proposition 1.3. (1) *For $p, q \in U$, the limit*

$$(7) \quad \Psi_{p,q}^{\Pi,\lambda}(X) = \lim_{R \rightarrow \infty} \exp \left(2(p - q) \int_{[-R, R] \cap U} \Pi(x, x) \lambda(x) dx \right) \prod_{x \in X: |x| \leq R} \left(\frac{x - p}{x - q} \right)^2$$

exists in $L_1(\text{Conf}(U), \mathbb{P}_{\Pi^q})$. Furthermore, for any compact subset $K \subset U$, there exists a subsequence $R_n \rightarrow \infty$, along which the almost sure convergence in (7) takes place for all $p, q \in K$.

(2) *There exists a positive function $\rho^{\Pi,\lambda} : U \rightarrow \mathbb{R}$ such that*

$$(8) \quad \int_{\text{Conf}(U)} \Psi_{p,q}^{\Pi,\lambda}(X) d\mathbb{P}_{\Pi^q}(X) = \frac{\rho^{\Pi,\lambda}(q) \Pi(p, p)}{\rho^{\Pi,\lambda}(p) \Pi(q, q)}.$$

If a configuration X is represented in the form $X = \{t_1, \dots, t_l\} \cup Y$, where $Y \in \text{Conf}(U)$, then, by definition, we have

$$\Psi_{p,q}^{\Pi,\lambda}(X) = \prod_{i=1}^l \left(\frac{t_i - p}{t_i - q} \right)^2 \Psi_{p,q}^{\Pi,\lambda}(Y).$$

We are now ready to formulate the analogue of Theorem 1.1.

Theorem 1.4. *Let $U \subset \mathbb{R}$ be an open set. Let $\Pi(x, y)$, $x, y \in U$, be a smooth kernel that induces an operator of orthogonal projection acting in $L_2(U, \text{Leb})$, satisfying Assumptions 1, 3 and such that the determinantal point process \mathbb{P}_Π is rigid. Let $I \subset U$ be a compact interval. Let $\lambda(x)$ be a continuous function on \mathbb{R} satisfying (6). Then*

1. *For almost any $2l$ distinct points $p_1, \dots, p_l, q_1, \dots, q_l \in U$, we have the $\mathbb{P}^{q_1, \dots, q_l}$ -almost sure equality*

$$\frac{d\mathbb{P}^{p_1, \dots, p_l}}{d\mathbb{P}^{q_1, \dots, q_l}}(X) = \frac{\det \Pi(q_i, q_j)|_{i,j=1, \dots, l}}{\det \Pi(p_i, p_j)|_{i,j=1, \dots, l}} \prod_{1 \leq i < j \leq l} \left(\frac{p_i - p_j}{q_i - q_j} \right)^2 \prod_{i=1}^l \frac{\rho^{\Pi,\lambda}(p_i)}{\rho^{\Pi,\lambda}(q_i)} \Psi_{p_i, q_i}^{\Pi,\lambda}(X).$$

2. *For \mathbb{P}_Π -almost every $X \in \text{Conf}(U)$, the measure $\mathbb{P}_\Pi(\cdot | X; U \setminus I)$ has the form*

$$(9) \quad Z(I, X, \lambda)^{-1} \prod_{1 \leq i < j \leq \#_I(X)} (t_i - t_j)^2 \prod_{i=1}^{\#_I(X)} \rho_{I,X}^{\Pi,\lambda}(t_i),$$

where $Z(I, X, \lambda)$ is the normalization constant and the function $\rho_{I,X}^{\Pi,\lambda}$ satisfies, for any $p, q \in I$, the relation

$$(10) \quad \frac{\rho_{I,X}^{\Pi,\lambda}(p)}{\rho_{I,X}^{\Pi,\lambda}(q)} = \frac{\rho^{\Pi,\lambda}(p)}{\rho^{\Pi,\lambda}(q)} \Psi_{p,q}^{\Pi,\lambda}(X|_{\mathbb{R} \setminus I}).$$

Remark. 1. The order of the points in Claim 1 is of course again immaterial: see the Remark to Theorem 1.1.

2. Different choices of the function λ result in the multiplication of $\Psi_{p,q}^{\Pi,\lambda}(X)$ by a constant. More precisely, given continuous functions λ_1 and λ_2 satisfying (6), the integral

$$\beta_\Pi(\lambda_1, \lambda_2) = \int_U (\lambda_1(x) - \lambda_2(x)) \Pi(x, x) dx$$

converges absolutely by Assumption 3. From the definitions we now have $\Psi_{p,q}^{\Pi,\lambda_1}(X) = \Psi_{p,q}^{\Pi,\lambda_2}(X) \exp(2(p - q)\beta_\Pi(\lambda_1, \lambda_2))$, and, consequently, we have $\frac{\rho^{\Pi,\lambda_1}(p)}{\rho^{\Pi,\lambda_1}(q)} = \frac{\rho^{\Pi,\lambda_2}(p)}{\rho^{\Pi,\lambda_2}(q)} \exp(2(q - p)\beta_\Pi(\lambda_1, \lambda_2))$. The expression (9) does not, of course, depend on the specific choice of λ .

3. Claim 2 of Theorem 1.4 implies that for \mathbb{P}_Π -almost every $X \in \text{Conf}(U)$ and any Borel automorphism F of U preserving the Lebesgue measure class and acting by the identity in the complement of a compact subset $V \subset U$, setting $X \cap V = \{p_1, \dots, p_l\}$ and keeping the same symbol F for the natural induced action of F on the space of configurations, we have

$$(11) \quad \frac{d\mathbb{P}_\Pi \circ F}{d\mathbb{P}}(X) = \prod_{1 \leq i < j \leq l} \left(\frac{F(p_i) - F(p_j)}{p_i - p_j} \right)^2 \prod_{i=1}^l \frac{\rho^{\Pi, \lambda}(F(p_i))}{\rho^{\Pi, \lambda}(p_i)} \frac{d\text{Leb} \circ F}{d\text{Leb}}(p_i) \Psi_{F(p_i), p_i}^{\Pi, \lambda}(X|_{U \setminus V}).$$

Let $\mathcal{S}(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)}$ be the sine-kernel. For $\lambda_0(x) = x/(x^2 + 1)$ (any odd function satisfying (6) would work), we have

$$(12) \quad \Psi_{p, q}^{\mathcal{S}, \lambda_0}(X) = \lim_{R \rightarrow \infty} \prod_{|x| \leq R} \left(\frac{x - p}{x - q} \right)^2.$$

Convergence in (12) is in L_1 and almost sure along a subsequence, for instance, $R_n = n^4$. Approximating the sine-kernel by Christoffel-Darboux kernels of Hermite polynomials in the usual way, we obtain $\rho^{\mathcal{S}, \lambda_0} = 1$. Theorem 1.4 now yields

Corollary 1.5. *Let I be a compact interval on \mathbb{R} . For $\mathbb{P}_\mathcal{S}$ -almost any configuration $X \in \text{Conf}(\mathbb{R})$, the conditional measure $\mathbb{P}_\mathcal{S}(\cdot | X; \mathbb{R} \setminus I)$ has the form*

$$(13) \quad Z(I, X)^{-1} \prod_{1 \leq i < j \leq \#I(X)} (t_i - t_j)^2 \prod_{i=1}^{\#I(X)} \rho_{I, X}^{\mathcal{S}}(t_i),$$

where $Z(I, X)$ is the normalization constant and the function $\rho_{I, X}^{\mathcal{S}}$ satisfies, for any $p, q \in I$, the relation

$$(14) \quad \frac{\rho_{I, X}^{\mathcal{S}}(p)}{\rho_{I, X}^{\mathcal{S}}(q)} = \lim_{R \rightarrow \infty} \prod_{x \in X \setminus I; |x| \leq R} \left(\frac{x - p}{x - q} \right)^2.$$

2. MULTIPLICATIVE FUNCTIONALS AND PALM MEASURES.

2.1. Proof of Proposition 1.3. Let $D_2\Pi$ stand for the Hessian of the kernel Π . The symbol $\|\cdot\|$ stands for the Euclidean norm of a vector or a matrix.

Lemma 2.1. *For any $\varepsilon > 0$ and compact subset $K \subset U$, there exists a positive constant $C(\varepsilon, K)$ such that for any $p, q \in K$ we have*

$$\sup_{R \in \mathbb{R}} \left| \int_{[-R, R] \cap U} \left(\left(\frac{x-p}{x-q} \right)^2 - 1 \right) \Pi^q(x, x) + 2(p-q) \Pi(x, x) \lambda(x) \right) dx \right| \leq \\ \leq C(\varepsilon, K) \left(1 + \max_{|x-q| \leq \varepsilon, |y-q| \leq \varepsilon} (||D_2 \Pi|| + ||\Pi||) + \int_U \frac{\Pi(x, x) dx}{1+x^2} \right).$$

The proof of Lemma 2.1 is routine. We represent the integral from $-R$ to R as a sum of two: first, the integral from $q - \varepsilon$ to $q + \varepsilon$, and, second, the integral over the remaining arcs. The first integral is estimated above by

$$C(\varepsilon, K) \max_{|x-q| \leq \varepsilon, |y-q| \leq \varepsilon} (||D_2 \Pi|| + ||\Pi||),$$

the second, in view of (6), by $C(\varepsilon, K) \int_U \frac{\Pi(x, x) dx}{1+x^2}$. The lemma is proved.

The result of [1] on the regularization of multiplicative functionals can be reformulated as follows:

Lemma 2.2. *For $p, q \in U$, the limit*

$$\lim_{R \rightarrow \infty} \exp \left(- \int_{[-R, R] \cap U} \left(\left(\frac{x-p}{x-q} \right)^2 - 1 \right) \Pi^q(x, x) dx \right) \prod_{x \in X: |x| \leq R} \left(\frac{x-p}{x-q} \right)^2$$

exists in $L_1(\text{Conf}(U), \mathbb{P}_{\Pi^q})$. Furthermore, for any compact subset K of U , there exists a subsequence $R_n \rightarrow \infty$, along which the almost sure convergence takes place for all $p, q \in K$.

Lemmata 2.1 and 2.2 imply Proposition 1.3.

2.2. The function $\rho^{\Pi, \lambda}$. By Proposition 1.3, we have $\Psi_{p,q}^{\Pi, \lambda}(X) \in L_1(\text{Conf}(U), \mathbb{P}_{\Pi^q})$. Assumption 1 implies the relation

$$L(p) = \frac{x-p}{x-q} L(q).$$

By Corollary 4.12 in [1], for any $p, q \in U$ there exists a positive constant $C_\lambda(p, q)$ such that for \mathbb{P}^q -almost every $X \in \text{Conf}(U)$ we have

$$(15) \quad \frac{d\mathbb{P}_{\Pi}^p}{d\mathbb{P}_{\Pi}^q}(X) = C_\lambda(p, q) \Psi_{pq}^{\Pi, \lambda}(X).$$

For $p, q, r \in U$, we have $\Psi_{pq}^{\Pi, \lambda} \Psi_{qr}^{\Pi, \lambda} = \Psi_{pr}^{\Pi, \lambda}$ and $C_\lambda(p, q) C_\lambda(q, r) = C_\lambda(p, r)$.

We now introduce a positive function $\rho^{\Pi, \lambda}$ on U by setting

$$C_\lambda(p, q) = \frac{\rho^{\Pi, \lambda}(p)\Pi(q, q)}{\rho^{\Pi, \lambda}(q)\Pi(p, p)},$$

and (8) is established. The function $\rho^{\Pi, \lambda}$ is of course defined up to a multiplicative constant.

In the case when the kernel Π satisfies the stronger assumption (2), we can simply take $\lambda = 0$ (even though $\lambda = 0$ does not satisfy (6)): the operator $(x - q)^{-1}\Pi^q$ belongs to the trace class (since so does $(x + i)^{-1}\Pi$), and we arrive at (2).

2.3. Relation between Radon-Nikodym derivatives of Palm measures of different orders. As before, let \mathbb{P} be a point process on a locally compact metric space E endowed with a sigma-finite measure μ without atoms. As usual, we assume that for any l the process \mathbb{P} admits the l -th correlation measure of the form $\rho_l(p_1, \dots, p_l)d\mu(p_1) \dots d\mu(p_l)$.

Proposition 2.3. *Assume that for any natural number l and $\mu^{\otimes l}$ -almost any two l -tuples $(p_1, \dots, p_l), (q_1, \dots, q_l)$ of distinct points in E , the reduced Palm measures $\mathbb{P}^{p_1, \dots, p_l}$ and $\mathbb{P}^{q_1, \dots, q_l}$ are equivalent. Then for $\mu^{\otimes 2l}$ -almost any $2l$ -tuple $(p_1, \dots, p_l, q_1, \dots, q_l)$ of distinct points in E we have*

$$\frac{\rho_l(p_1, \dots, p_l)d\mathbb{P}^{p_1, \dots, p_l}}{\rho_l(q_1, \dots, q_l)d\mathbb{P}^{q_1, \dots, q_l}}(X) = \prod_{i=1}^l \frac{\rho_1(p_i)}{\rho_1(q_i)} \cdot \frac{d\mathbb{P}^{p_i}}{d\mathbb{P}^{q_i}}(X \cup q_1 \cup \dots \cup q_{i-1} \cup p_{i+1} \cup \dots \cup p_l).$$

Proof. For μ -almost any distinct $p, q, r_1, \dots, r_m \in E$, we clearly have

$$\frac{\rho_{m+1}(p, r_1, \dots, r_m)}{\rho_{m+1}(q, r_1, \dots, r_m)} \frac{d\mathbb{P}^{p, r_1, \dots, r_m}}{d\mathbb{P}^{q, r_1, \dots, r_m}}(X) = \frac{\rho_1(p)}{\rho_1(q)} \cdot \frac{d\mathbb{P}^p}{d\mathbb{P}^q}(X \cup r_1 \cup \dots \cup r_m).$$

The proposition is now proved by induction. For $l = 2$ and μ -almost any p_1, p_2, q_1, q_2 , we have

$$\begin{aligned} \frac{\rho_2(p_1, p_2)}{\rho_2(q_1, q_2)} \frac{d\mathbb{P}^{p_1, p_2}}{d\mathbb{P}^{q_1, q_2}}(X) &= \frac{\rho_2(p_1, p_2)}{\rho_2(q_1, p_2)} \frac{d\mathbb{P}^{p_1, p_2}}{d\mathbb{P}^{q_1, p_2}}(X) \cdot \frac{\rho_2(q_1, p_2)}{\rho_2(q_1, q_2)} \frac{d\mathbb{P}^{q_1, p_2}}{d\mathbb{P}^{q_1, q_2}}(X) = \\ &= \frac{\rho_1(p_1)}{\rho_1(q_1)} \frac{d\mathbb{P}^{p_1}}{d\mathbb{P}^{q_1}}(X \cup p_2) \cdot \frac{\rho_1(p_2)}{\rho_1(q_2)} \frac{d\mathbb{P}^{p_2}}{d\mathbb{P}^{q_2}}(X \cup q_1). \end{aligned}$$

For the induction step, we write

$$\frac{\rho_l(p_1, \dots, p_l)d\mathbb{P}^{p_1, \dots, p_l}}{\rho_l(q_1, \dots, q_{l-1}, p_l)d\mathbb{P}^{q_1, \dots, q_{l-1}, p_l}}(X) = \frac{\rho_l(p_1, \dots, p_{l-1})d\mathbb{P}^{p_1, \dots, p_{l-1}}}{\rho_l(q_1, \dots, q_{l-1})d\mathbb{P}^{q_1, \dots, q_{l-1}}}(X \cup p_l),$$

whence, using the induction hypothesis, we conclude

$$\frac{\rho_l(p_1, \dots, p_l)d\mathbb{P}^{p_1, \dots, p_l}}{\rho_l(q_1, \dots, q_l)d\mathbb{P}^{q_1, \dots, q_l}}(X) =$$

$$\begin{aligned}
&= \frac{\rho_l(p_1, \dots, p_l) d\mathbb{P}^{p_1, \dots, p_l}}{\rho_l(q_1, \dots, q_{l-1}, p_l) d\mathbb{P}^{q_1, \dots, q_{l-1}, p_l}}(X) \cdot \frac{\rho_l(q_1, \dots, q_{l-1}, p_l) d\mathbb{P}^{q_1, \dots, q_{l-1}, p_l}}{\rho_l(q_1, \dots, q_l) d\mathbb{P}^{q_1, \dots, q_l}}(X) = \\
&= \prod_{i=1}^{l-1} \frac{\rho_1(p_i)}{\rho_1(q_i)} \cdot \frac{d\mathbb{P}^{p_i}}{d\mathbb{P}^{q_i}}(X \cup q_1 \cup \dots \cup q_{i-1} \cup p_{i+1} \cup \dots \cup p_l) \times \frac{\rho_1(p_l)}{\rho_1(q_l)} \cdot \frac{d\mathbb{P}^{p_l}}{d\mathbb{P}^{q_l}}(X \cup q_1 \cup \dots \cup q_{l-1}).
\end{aligned}$$

The proposition is proved completely.

Corollary 2.4. *Let \mathbb{P} be a point process satisfying all assumptions of Proposition 2.3. If there exists a positive Borel function $\Psi : E \times E \times \text{Conf}(E) \rightarrow \mathbb{R}_+$ and a positive Borel function $\Phi : \text{Conf}_2(E) \rightarrow \mathbb{R}_+$ such that*

- (1) *for μ -almost any $p, q \in E$, for \mathbb{P}^q -almost any $X \in \text{Conf}(E)$, any $l \in \mathbb{N}$ and any distinct particles $r_1, \dots, r_l \in X$, we have*

(16)

$$\Psi(p, q, X) = \frac{\Phi(p, r_1)}{\Phi(q, r_1)} \cdot \frac{\Phi(p, r_2)}{\Phi(q, r_2)} \cdot \dots \cdot \frac{\Phi(p, r_l)}{\Phi(q, r_l)} \times \Psi(p, q, X \setminus \{r_1, \dots, r_l\});$$

- (2) *for any $p, q, r \in E$ and \mathbb{P}^q -almost any $X \in \text{Conf}(E)$ we have*

$$\Psi(p, q, X) \cdot \Psi(q, r, X) = \Psi(p, r, X);$$

- (3) *for μ -almost any $p, q \in E$ and \mathbb{P}^q -almost any $X \in \text{Conf}(E)$ we have*

$$\frac{\rho_1(p) d\mathbb{P}^p}{\rho_1(q) d\mathbb{P}^q}(X) = \Psi(p, q, X),$$

then, for $\mu^{\otimes l}$ -almost any $(p_1, \dots, p_l) \in \text{Conf}_l(E)$, $(q_1, \dots, q_l) \in \text{Conf}_l(E)$ and $\mathbb{P}^{q_1, \dots, q_l}$ -almost any $X \in \text{Conf}(E)$, we have

$$\frac{\rho_l(p_1, \dots, p_l) d\mathbb{P}^{p_1, \dots, p_l}}{\rho_l(q_1, \dots, q_l) d\mathbb{P}^{q_1, \dots, q_l}}(X) = \prod_{1 \leq i < j \leq l} \frac{\Phi(p_i, p_j)}{\Phi(q_i, q_j)} \cdot \prod_{i=1}^l \Psi(p_i, q_i, X).$$

Proposition 8, together with Proposition 2.3 and Corollary 2.4, applied to our functional $\Psi_{p,q}^{\Pi, \lambda}$ satisfying (16) with $\Phi(p, q) = |p - q|^2$, directly implies the first claim of Theorems 1.1, 1.4. We proceed to proving the second one.

2.4. Conditional Campbell measures. The following Proposition 2.5 will not be used in the proof and is included to clarify the context.

Let \mathbb{P} be a point process with locally finite intensity (in other words, admitting the first correlation measure) on E . Write $\xi\mathbb{P}$ for the first correlation measure of \mathbb{P} . Let $C \subset E$ be a Borel subset. Let $\overline{\mathbb{P}}_C$ stand for the image of \mathbb{P} under the natural projection map $\pi_C : \text{Conf}(E) \rightarrow \text{Conf}(C)$.

Proposition 2.5. *Assume that for \mathbb{P} -almost every X the intensity $\xi\mathbb{P}_{(\cdot|X;C)}$ of the conditional process is absolutely continuous with respect to $\xi\mathbb{P}$. Then*

- (1) *for $\xi\mathbb{P}$ -almost every $q \in E$ and $\overline{\mathbb{P}}_C$ -almost every $Y \in \text{Conf}(C)$ we have $(\mathbb{P}^q)_{(\cdot|Y;C)} = (\mathbb{P}_{(\cdot|Y;C)})^q$;*

(2) for $\xi\mathbb{P}$ -almost every $q \in E$ we have

$$(17) \quad \mathbb{P}^q = \int_{\text{Conf}(C)} \mathbb{P}_{(\cdot|Y;C)}^q \cdot \frac{d\xi\mathbb{P}_{(\cdot|Y;C)}}{d\xi\mathbb{P}}(q) \cdot d\overline{\mathbb{P}}_C(Y).$$

Proof. Recall that the Campbell measure $\mathcal{C}_{\mathbb{P}}$ of the point process \mathbb{P} is defined, for a compact subset $B \subset E$ and a Borel subset $Z \subset \text{Conf}(E)$, by the formula

$$\mathcal{C}_{\mathbb{P}}(B \times Z) = \int_Z \#_B(X) \cdot d\mathbb{P}(X).$$

By definition, we have $\mathcal{C}_{\mathbb{P}} = \int_{\text{Conf}(C)} \mathcal{C}_{\mathbb{P}_{(\cdot|Y;C)}} d\overline{\mathbb{P}}_C(Y)$. Let $\check{\mathbb{P}}^q$ stand for the

non-reduced Palm measure of \mathbb{P} at the point q . We have $\mathcal{C}_{\mathbb{P}} = \int_E \check{\mathbb{P}}^q d\xi\mathbb{P}(q)$

and, similarly, $\mathbb{P}_{(\cdot|Y;C)} = \int_E \check{\mathbb{P}}_{(\cdot|Y;C)}^q d\xi\mathbb{P}_{(\cdot|Y;C)}(q)$. Removing the point at q

and passing to reduced Palm measures, we arrive at (17).

Corollary 2.6. *Let \mathbb{P} be a point process on E such that for \mathbb{P} -almost every X the intensity $\xi\mathbb{P}_{(\cdot|X;C)}$ of the conditional process is absolutely continuous with respect to $\xi\mathbb{P}$ and for $\xi\mathbb{P}$ -almost any $p, q \in E$ the reduced Palm measures \mathbb{P}^p and \mathbb{P}^q are equivalent. Then for $\xi\mathbb{P}$ -almost any $p, q \in E$ and \mathbb{P} -almost any $X \in \text{Conf}(E)$ we have*

$$(18) \quad \frac{d\mathbb{P}^p}{d\mathbb{P}^q}(X) = \frac{\frac{d\xi\mathbb{P}_{(\cdot|X;C)}}{d\xi\mathbb{P}}(p)}{\frac{d\xi\mathbb{P}_{(\cdot|X;C)}}{d\xi\mathbb{P}}(q)} \cdot \frac{d\mathbb{P}^p_{(\cdot|X;C)}}{d\mathbb{P}^q_{(\cdot|X;C)}}(X|_{E \setminus C}).$$

Corollary 2.6 is insufficient for our purposes: we need relation (18) to hold on a fixed subset of $\text{Conf}(E)$ of full measure and for $\xi\mathbb{P}$ -almost any $p, q \in E$. To check this, we use the quasi-invariance of our point processes under the group of compactly supported piecewise isometries of E .

3. PALM MEASURES AND CONDITIONAL MEASURES.

3.1. Characterization of conditional measures. In this subsection, a general result is formulated describing conditional measures of point processes in terms of Radon-Nikodym derivatives of Palm measures of the same order.

Let E be an open subset of \mathbb{R}^d , endowed with the Lebesgue measure $dv = dv_1 \dots dv_d$. Let \mathbb{P} be a point process on E satisfying the following.

Assumption 4. *The point process \mathbb{P} admits correlation measures of all orders. For any $l > 0$, the l -th correlation measure of \mathbb{P} has the form*

$$\rho_l(p_1, \dots, p_l) dp_1 \dots dp_l,$$

where ρ_l is a symmetric continuous function on E^l .

Recall that the tail sigma-algebra on $\text{Conf}(E)$ is the intersection of all sigma-algebras $\mathcal{F}_{E \setminus B}$ over all compact $B \subset E$.

Assumption 5. *There exists a Borel subset $\mathcal{W} \subset \text{Conf}(E)$, belonging to the tail sigma-algebra of $\text{Conf}(E)$, and, for any $l > 0$, a Borel measurable function $\Psi(p_1, \dots, p_l; q_1, \dots, q_l; X)$, defined for $X \in \mathcal{W}$ and any two distinct l -tuples of points not containing particles of the configuration X , such that the following holds:*

- (1) $\mathbb{P}(\mathcal{W}) = 1$;
- (2) for fixed X , the function $\Psi(p_1, \dots, p_l; q_1, \dots, q_l; X)$ is continuous in $(p_1, \dots, p_l) \in \text{Conf}_l(E \setminus X)$, $(q_1, \dots, q_l) \in \text{Conf}_l(E \setminus X)$;
- (3) for fixed X and any three l -tuples (p_1, \dots, p_l) , (q_1, \dots, q_l) , (r_1, \dots, r_l) in $\text{Conf}_l(E \setminus X)$, we have

$$\Psi(p_1, \dots, p_l; q_1, \dots, q_l; X) = \Psi(p_1, \dots, p_l; r_1, \dots, r_l; X) \Psi(r_1, \dots, r_l; q_1, \dots, q_l; X).$$

- (4) for \mathbb{P} -almost any $Y \in \mathcal{W}$, any l distinct particles $(p_1, \dots, p_l) \in Y$ and $\mu^{\otimes l}$ -almost any l -tuple $(q_1, \dots, q_l) \in \text{Conf}_l(E \setminus Y)$, we have

$$\frac{d\mathbb{P}^{p_1, \dots, p_l}}{d\mathbb{P}^{q_1, \dots, q_l}}(Y) = \Psi(p_1, \dots, p_l; q_1, \dots, q_l; Y \setminus \{p_1, \dots, p_l\}).$$

Proposition 3.1. *Let \mathbb{P} be a rigid point process on E satisfying Assumptions 4, 5. Let $I \subset E$ be a precompact open subset. Let $l \in \mathbb{N}$ be such that*

$$\mathbb{P}(\{X : \#_I(X) = l\}) > 0.$$

Then, for \mathbb{P} -almost every $X \in \text{Conf}(E)$ such that $\#_I(X) = l$ and almost any distinct points $q_1, \dots, q_l \in E$, the conditional measure $\mathbb{P}(\cdot | X, E \setminus I)$ has the form

$$(19) \quad Z_{q_1, \dots, q_l}^{-1} \Psi(p_1, \dots, p_l; q_1, \dots, q_l; X |_{E \setminus I}) \rho_l(p_1, \dots, p_l) dp_1 \dots dp_l,$$

where Z_{q_1, \dots, q_l} is the normalization constant.

Remark. The reference l -tuple $q_1, \dots, q_l \in E$ can be chosen arbitrarily; a different choice results in a change of the normalization constant.

3.2. Quasi-invariance under piecewise isometries. We endow \mathbb{R}^d with the norm $\|v\| = \max_{i=1, \dots, d} |v_i|$ and the corresponding metric. The balls in this metric are cubes. We take distinct points $p_1, \dots, p_l, q_1, \dots, q_l \in E$, take $\delta_1 > 0, \delta_2 > 0, \dots, \delta_l > 0$ sufficiently small in such a way that the balls of radius δ_i centred at $p_1, \dots, p_l, q_1, \dots, q_l$ do not intersect, and

consider the piecewise isometry of E that sends the closed ball of radius δ_i centred at p_i to the corresponding ball centred at q_i , $i = 1, \dots, l$, leaving the complement to the union of the closed balls fixed. The group generated by such piecewise isometries is denoted $\mathfrak{G} = \mathfrak{G}(E)$. For example, if $E = \mathbb{R}$, then the resulting group is the group of all interval exchange transformations on \mathbb{R} , while in higher dimension we arrive at the group of cube exchanges. The countable subgroup $\mathfrak{G}_0 = \mathfrak{G}_0(E)$ generated by transformations of the above form such that the centres of all the balls have rational coordinates and the radii of the balls are rational. For a subset $C \subset E$, let $\mathfrak{G}(C)$ and $\mathfrak{G}_0(C)$ be the subgroups of maps acting as the identity on $E \setminus C$. For brevity, we write $\mathbf{p} = (p_1, \dots, p_l)$, $d\mathbf{p} = dp_1 \dots dp_l$, $T\mathbf{p} = (Tp_1, \dots, Tp_l)$, etc.

Proposition 3.2. *Let $I \subset \mathbb{R}^d$ be a bounded open set, let $l \in \mathbb{N}$. Let $F : \text{Conf}_l(I) \rightarrow \mathbb{R}_+$ be a positive continuous function. Let μ be a Borel probability measure on $\text{Conf}_l(I)$ such that the equality*

$$(20) \quad \frac{d\mu \circ T}{d\mu}(\mathbf{p}) = \frac{F(T\mathbf{p})}{F(\mathbf{p})}.$$

holds μ -almost surely for all $T \in \mathfrak{G}_0(I)$. Then (20) holds for all $T \in \mathfrak{G}(I)$ and $d\mu(\mathbf{p}) = F(\mathbf{p})d\mathbf{p}$.

Proof. We first show that μ assigns mass zero to boundaries of balls:

Lemma 3.3. *For any $p \in I$ we have $\mu(\{\mathbf{r} \in \text{Conf}_l(I) : p \in \mathbf{r}\}) = 0$.*

Remark. The continuity of F is essential, since any atomic measure with atoms of positive mass at all rational points in $\text{Conf}_l(I)$ is quasi-invariant under $\mathfrak{G}_0(I)$.

Proof of Lemma 3.3. First, we note that the measure μ cannot have atoms: if $\mu(\mathbf{p}) = \delta_0 > 0$, then, since the orbit of the configuration \mathbf{p} under \mathfrak{G}_0 is dense in $\text{Conf}_l(I)$ and (20) implies that there exists $\delta_1 > 0$ depending on δ_0 and F such that the set $\{\mathbf{q} \in \text{Conf}_l(I) : \mu(\mathbf{q}) \geq \delta_1\}$ is infinite; but then the measure μ cannot be finite. Next, for any $i \leq d$ and any distinct points $p_1, \dots, p_i \in I$ we show

$$\mu(\{\mathbf{r} \in \text{Conf}_l(I) : p_1, \dots, p_i \in \mathbf{r}\}) = 0.$$

We argue by induction on $i = d, d-1, \dots, 1$. The case $i = d$ is precisely the absence of atoms already established. For the induction step, assume $\mu(\{\mathbf{r} : p_1, \dots, p_i \in \mathbf{r}\}) > 0$. Then there exist points $q_1, \dots, q_i \in I$ and $\delta > 0$, $\varepsilon > 0$ and a ball $B(\varepsilon)$ of radius ε in $\text{Conf}_l(I)$ such that distances between distinct q_k all exceed 2ε and we have $\mu(\{\mathbf{r} : q_1, \dots, q_i \in \mathbf{r}\} \cap B(\varepsilon)) > \delta$. Write $D = \{\mathbf{r} : q_1, \dots, q_i \in \mathbf{r}\} \cap B(\varepsilon)$. By continuity of F , there exists $\delta_1 > 0$ such that the set of the “shifts” TD of the set D by elements $T \in \mathfrak{G}_0(I)$ satisfying $\mu(TD) > \delta_1$ is infinite. The induction

hypothesis implies $\mu(D \cap TD) = 0$. It follows that the measure μ cannot be finite, and Lemma 3.3 is proved completely.

We proceed with the proof of Proposition 3.2. A ball of radius r centred at a configuration $\mathbf{p} \in \text{Conf}_l(I)$ will be called *proper* if the distances between the distinct p_i are all less than $r/2$.

Take two finite collections $B_1, \dots, B_k, B'_1, \dots, B'_k$ of disjoint isometric proper balls and let T be a piecewise isometry interchanging B_i and B'_i , $i = 1, \dots, k$. To establish Proposition 3.2, it suffices to establish (20) for piecewise isometries T of this form.

Take an exhausting sequence $B_{n,i} \subset B_i, B'_{n,i} \subset B'_i$ of isometric balls with rational centres and radii. Define $T_n \in \mathfrak{G}_0$ as the map that interchanges $B_{n,i}$ and $B'_{n,i}$, $i = 1, \dots, k$. Lemma 3.3 implies that the sequence $\mu \circ T_n$ weakly converges to $\mu \circ T$ and also that the sequence $F(T_n \mathbf{p})\mu$ weakly converges to the limit $F(T \mathbf{p})\mu$ as $n \rightarrow \infty$.

Take $\varepsilon > 0$. Set $\text{Conf}_{l,\varepsilon}(I) = \{\mathbf{p} \in \text{Conf}_l(I) : \min_{i,j=1,\dots,l} |p_i - p_j| \geq \varepsilon\}$. Let φ be a bounded continuous function on $\text{Conf}_l(I)$ supported on $\text{Conf}_{l,\varepsilon}(I)$. The function $\varphi(\mathbf{p})/F(\mathbf{p})$ is then bounded and continuous, and we have

$$\lim_{n \rightarrow \infty} \int_{\text{Conf}_l(I)} \varphi(\mathbf{p}) \cdot \frac{F(T_n \mathbf{p})}{F(\mathbf{p})} d\mu(\mathbf{p}) = \int_{\text{Conf}_l(I)} \varphi(\mathbf{p}) \cdot \frac{F(T \mathbf{p})}{F(\mathbf{p})} d\mu(\mathbf{p}),$$

whence the sequence of probability measures $\frac{F(T_n \mathbf{p})}{F(\mathbf{p})}\mu = \mu \circ T_n$ vaguely converges, as $n \rightarrow \infty$, to the measure $\frac{F(T \mathbf{p})}{F(\mathbf{p})}\mu$. Since the sequence $\mu \circ T_n$ weakly converges to $\mu \circ T$, the equality (20) is proved for all $T \in \mathfrak{G}$.

To conclude the proof of Proposition 3.2, consider the measure η given by $d\eta(\mathbf{p}) = d\mu(\mathbf{p})/F(\mathbf{p})$. By continuity and positivity of F , for any $\varepsilon > 0$, the measure η is finite in restriction to $\text{Conf}_{l,\varepsilon}(I)$. Since η is \mathfrak{G} -invariant, the measure η , in restriction to $\text{Conf}_{l,\varepsilon}(I)$, coincides with the Lebesgue measure. Since ε is arbitrary, Proposition 3.2 is proved completely.

3.3. Completion of the proof of Proposition 3.1. Let S be a standard Borel space, let μ be a Borel probability measure on S . Let \mathcal{F} be a σ -algebra of Borel subsets of S , let π be the corresponding measurable partition. We let $\bar{\mu}$ be the quotient measure of μ under the partition π , and, for an element ξ of the partition π , we let μ^ξ be the corresponding conditional measure. Finally, let T be a Borel transformation of the space S such that every set of \mathcal{F} is T -invariant and the measure μ is T -quasi-invariant. The definitions directly imply

Proposition 3.4. *Let F be a Borel function such that the equality*

$$\frac{d\mu \circ T}{d\mu} = F$$

holds μ -almost surely. Then for $\bar{\mu}$ -almost every element ξ of the partition π we have the μ^ξ -almost sure equality

$$\frac{d\mu^\xi \circ T}{d\mu^\xi} = F.$$

Proposition 2.9 in [1] claims that for a piecewise isometry $T \in \mathfrak{G}$ acting as the identity beyond a compact set V and a configuration $X \in \text{Conf}(E)$ such that $X \cap V = \{p_1, \dots, p_l\}$, we have, \mathbb{P} -almost surely, the equality

$$(21) \quad \frac{d\mathbb{P} \circ T}{d\mathbb{P}}(X) = \frac{\rho_l(Tp_1, \dots, Tp_l)}{\rho_l(p_1, \dots, p_l)} \frac{d\mathbb{P}^{Tp_1, \dots, Tp_l}}{d\mathbb{P}^{p_1, \dots, p_l}}(X \setminus \{p_1, \dots, p_l\}).$$

Let $I \subset E$ be precompact and open. By Proposition 3.4, for \mathbb{P} -almost any $X \in \text{Conf}(E)$ and any $T \in \mathfrak{G}_0$, the measure $\mathbb{P}(\cdot | X, E \setminus I)$ satisfies the equality (21) (in which one must, of course, substitute $\mathbb{P}(\cdot | X, E \setminus I)$ for \mathbb{P}). By Proposition 3.2, the same equality holds for all $T \in \mathfrak{G}$ and the measure $\mathbb{P}(\cdot | X, E \setminus I)$ has the form (19). Proposition 3.1 is proved completely.

4. CONTINUITY OF THE FUNCTIONS ρ^Π , $\rho^{\Pi, \lambda}$ AND THE PROOFS OF PROPOSITION 1.2, COROLLARY 1.5.

4.1. The trace class case. Let $D_1\Pi$ stand for the Jacobi matrix of the kernel Π . Our definitions immediately imply the following important continuity property of the function ρ^Π .

Proposition 4.1. *Let Π_n be a sequence of smooth kernels, each inducing an operator of orthogonal projection in $L_2(U, \text{Leb})$, each satisfying Assumptions 1 and 2. Assume that, as $n \rightarrow \infty$, we have*

- (1) $\Pi_n \rightarrow \Pi, D_1\Pi_n \rightarrow D_1\Pi, D_2\Pi_n \rightarrow D_2\Pi$ uniformly on compact subsets of $U \times U$;
- (2) $(|x| + 1)^{-1}\Pi_n \rightarrow (|x| + 1)^{-1}\Pi$ in the space of trace class operators acting in $L_2(U, \text{Leb})$.

Then, for any $p, q \in U$, we have $\lim_{n \rightarrow \infty} \frac{\rho^{\Pi_n}(p)}{\rho^{\Pi_n}(q)} = \frac{\rho^\Pi(p)}{\rho^\Pi(q)}$.

4.2. The Bessel kernel: computation of the function ρ^{J^s} . Let $s > -1$. Let $P_n^{(s)}$ be the standard Jacobi orthogonal polynomials corresponding to the weight $(1-u)^s$. Let $\tilde{K}_n^{(s)}(u_1, u_2)$ the n -th Christoffel-Darboux kernel of the Jacobi orthogonal polynomial ensemble. Recall that the classical Heine-Mehler asymptotics for Jacobi orthogonal polynomials (see e.g. Chapter 8 in Szegő [11]) implies that for any $s > -1$, as $n \rightarrow \infty$, the kernel $\tilde{K}_n^{(s)}$

converges to the kernel J_s uniformly in the totality of variables on compact subsets of $(0, +\infty) \times (0, +\infty)$, indeed, on arbitrary simply connected compact subsets of $(\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0)$. Our next aim is to justify the limit transition

$$(22) \quad \lim_{n \rightarrow \infty} \frac{\rho^{\Pi_n}(p)}{\rho^{\Pi_n}(q)} = \lim_{n \rightarrow \infty} \frac{(1 - (1 - p/2n^2))^s}{(1 - (1 - q/2n^2))^s} = \frac{p^s}{q^s} = \frac{\rho^{J_s}(p)}{\rho^{J_s}(q)}.$$

By Proposition 4.1, it remains to prove that $(1+x)^{-1} \tilde{K}_n^{(s)} \rightarrow (1+x)^{-1} J_s$ in the space of trace class operators acting in $L_2(\mathbb{R}_+)$. For $s > 0$, this trace class convergence directly follows from standard inequalities for Jacobi polynomials, see as e.g. Theorem 7.3.2 in Szegő [11]. To treat the case $s \in (-1, 0]$, note that for any $s > -1$ we have the recurrence relations

$$(23) \quad \tilde{K}_n^{(s)}(u_1, u_2) = \frac{s+1}{2^{s+1}} P_{n-1}^{(s+1)}(u_1)(1-u_1)^{s/2} P_{n-1}^{(s+1)}(u_2)(1-u_2)^{s/2} + \tilde{K}_n^{(s+2)}(u_1, u_2)$$

$$(24) \quad J_s(x, y) = J_{s+2}(x, y) + \frac{s+1}{\sqrt{xy}} J_{s+1}(\sqrt{x}) J_{s+1}(\sqrt{y}).$$

Relations (23), (24) imply the convergence $(1+x)^{-1} \tilde{K}_n^{(s)} \rightarrow (1+x)^{-1} J_s$ in trace class norm for any $s > -1$. Proposition 1.2 is proved completely.

4.3. The Hilbert-Schmidt Case. Our definitions directly imply

Proposition 4.2. *Let Π_n be a sequence of smooth kernels, each inducing an operator of orthogonal projection acting in $L_2(U, \text{Leb})$, each satisfying Assumptions 1 and 3. If, as $n \rightarrow \infty$, we have*

- (1) $\Pi_n \rightarrow \Pi, D_1 \Pi_n \rightarrow D_1 \Pi, D_2 \Pi_n \rightarrow D_2 \Pi$ uniformly on compact subsets of $U \times U$;
- (2) $(x+i)^{-1} \Pi_n \rightarrow (x+i)^{-1} \Pi$ in the Hilbert-Schmidt norm,

then, for any continuous λ satisfying (6) and any $p, q \in U$ we have

$$(25) \quad \lim_{n \rightarrow \infty} \frac{\rho_{\Pi_n}^\lambda(p)}{\rho_{\Pi_n}^\lambda(q)} = \frac{\rho_\Pi^\lambda(p)}{\rho_\Pi^\lambda(q)}.$$

Proposition 4.3. *If $\Pi_n \rightarrow \Pi$ uniformly on compact subsets of U and there exists $\alpha, 0 \leq \alpha < 1/2$, such that*

$$(26) \quad \sup_{n \in \mathbb{N}, x \in \mathbb{R}} \frac{\Pi_n(x, x)}{1 + |x|^\alpha} < +\infty.$$

Then $(x+i)^{-1} \Pi_n \rightarrow (x+i)^{-1} \Pi$ in Hilbert-Schmidt norm.

Proof. Indeed, by Grümme's theorem (see e.g. Simon [9]), it suffices to check the relation

$$(27) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\Pi_n(x, x) dx}{1 + x^2} = \int_{-\infty}^{\infty} \frac{\Pi(x, x) dx}{1 + x^2}.$$

For any $R_0 > 0$, the uniform convergence of our kernels on compact subsets implies the convergence

$$\lim_{n \rightarrow \infty} \int_{-R_0}^{R_0} \frac{\Pi_n(x, x) dx}{1 + x^2} = \int_{-R_0}^{R_0} \frac{\Pi(x, x) dx}{1 + x^2}.$$

Condition (26), in turn, immediately implies, for any $\varepsilon > 0$, the existence of $R_0 > 0$ such that

$$(28) \quad \sup_{n \in \mathbb{N}} \int_{|x| > R_0} \frac{\Pi_n(x, x)}{1 + x^2} < +\infty,$$

convergence (27) follows, and Proposition 4.3 is proved.

4.4. The sine-kernel. Let $\lambda_0(x) = x(x^2 + 1)^{-1}$ so that (12) holds. Since

$$\text{Var}_{\mathbb{P}_{\mathcal{S}}} \left(\sum_{x \in X, |x| \geq R} (\log |x - p| - \log |x - q|) \right) = O(R^{-2}).$$

the Borel-Cantelli lemma implies convergence in (12), for example, along the sequence $R_n = n^4$. Let $\tilde{K}_n^{(H)}$ be the Christoffel-Darboux kernel of the standard Hermite polynomials and set

$$K_n^{(H)}(x, y) = \frac{\pi}{\sqrt{2n}} \tilde{K}_n^{(H)}\left(\frac{x}{\sqrt{2n}}, \frac{y}{\sqrt{2n}}\right).$$

We have $\lim_{n \rightarrow \infty} K_n^{(H)}(x, y) = \mathcal{S}(x, y)$. Convergence is uniform with all derivatives as long as x, y range over compact subsets of the complex plane. The Plancherel-Rotach asymptotic for Hermite polynomials, see e.g. Theorem 8.22.9 in Szegő [11], implies (26) for $\Pi_n = K_n^{(H)}$, and Proposition 4.3 implies the Hilbert-Schmidt convergence $(x + i)^{-1} K_n^{(H)} \rightarrow (x + i)^{-1} \mathcal{S}$.

Since λ_0 is odd and $K_n^{(H)}(x, x)$ is even, similarly to (12), we have

$$\Psi_{p,q}^{K_n^{(H)}, \lambda_0}(x_1, \dots, x_n) = \prod_{i=1}^n \left(\frac{x_i - p}{x_i - q} \right)^2.$$

Since $\lim_{n \rightarrow \infty} \exp(-p^2/2n + q^2/2n) = 1$, we conclude $\rho^{\mathcal{S}, \lambda_0}(p) = 1$.

Remark. The Airy kernel satisfies all assumptions of Theorem 1.4; the explicit constants will be given in the sequel to this paper.

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AIX-MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373

THE STEKLOV INSTITUTE OF MATHEMATICS, MOSCOW

THE INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, MOSCOW

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, MOSCOW